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Kernels in Perfect Line-Graphs

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A kernel of a directed graph D is a set of vertices which is both independent and absorbant. In 1983, Berge and Duchet conjectured that an undirected graph G is perfect if and only if the following condition is satisfied: "If D is any orientation of G such that every clique of D has a kernel, then D has a kernel." We prove here that the conjecture holds when G is the line-graph of another graph H, i.e., G represents the incidence between the edges of H. © 1992 Academic Press, Inc.

1. INTRODUCTION

The graphs considered here have no loops, but they may have multiple edges. Unless otherwise specified, we use the standard terminology of Berge [1]. A directed graph D can be viewed as a given orientation of its underlying undirected subgraph G. In the digraphs considered here, pairs of opposite arcs (i.e., directed cycles of length 2) are permitted. However, we will consider that, in the underlying undirected subgraph G of D, such a pair of opposite arcs corresponds to just one edge, not two, so that G is a simple graph. We will then say that this edge of G is *symmetrically directed* in D.

Given any arc from a vertex x to a vertex y, one says that y is a *successor* of x. A subset K of vertices of a digraph D is called *absorbant* if every vertex outside K has a successor in K. A subset of vertices is called *independent* if any two of its vertices are non-adjacent. A *kernel* of a digraph D is a subset of vertices that is both absorbant and independent.

Note that a kernel of a directed clique C is simply a *sink*, i.e., a vertex s of C that is a successor of every vertex of $C-s$. Likewise, a vertex t of C such that every vertex of $C-t$ is a successor of t wil be called a *source* of the clique C. A directed graph D in which every clique has a sink—or, equivalently, every clique has a source—is called a *normal orientation* of its underlying undirected graph. An undirected graph G is then said to be

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solvable ([6], or *nearly perfect* in [3]) if every normal orientation of G has a kernel.

A graph G is *perfect* if the vertices of any induced subgraph F of G can be colored with a number of colors equal to the size of a maximum clique of F. Equivalently, G is perfect if every induced subgraph F contains an independent set that meets all maximum cliques of F. See [2] for more information on perfect graphs.

In 1983, Berge and Duchet [3] formulated the following two conjectures.

Conjecture A. Let D be a normal orientation of a perfect graph G. Then D has a kernel.

Conjecture B. A graph is perfect if and only if it is solvable.

The problem of the existence of a kernel in a digraph is a difficult one (see, for example, [9, 13, 16]), and this can explain why Conjecture A is settled only for a few special classes of perfect graphs like *i*-triangulated graphs [19] and the complements of strongly perfect graphs [10]. A weaker form of Conjecture A is known to hold for comparability graphs [12, 18] and parity graphs [7, 14] and is settled for Meyniel graphs in [11]. See [6, 8, 18] for more information; a recent survey is [4].

We will show here that Conjecture A holds for any *line-graph*. If H is an undirected graph (possibly with multiple edges), the line-graph $L(H)$ of H is defined as the graph whose vertices represent the edges of H, so that two vertices of $L(H)$ are adjacent if and only if they represent two edges that are incident in H. Note that H may have multiple edges; but $L(H)$ does not.

2. The Main Result

THEOREM 1. *A line-graph is perfect if and only if it is solvable.*

In the proof of this result, we will use the characterization of perfect line-graphs that is given in Theorem 2. Perfect line-graphs have been studied previously by Trotter [20]. We give here a more extensive characterization.

THEOREM 2. *Let $G = L(H)$ be the line-graph of a graph H. Then the following three conditions are equivalent*:

(a) *G is a perfect graph.*

(b) *H does not contain any odd cycle of length at least* 5.

(c) *Any connected partial subgraph H' of H satisfies at least one of the following properties*:

(i) *H' is a bipartite graph;*

(ii) *H' is a clique with four vertices;*

(iii) *H' consists of exactly $p+2$ vertices $x_1, ..., x_p$, a, b, such that $\{x_1, ..., x_p\}$ is an independent set, and $\{x_j, a, b\}$ is a clique for each $j = 1, ..., p$.*

(iv) *H' has a cut-vertex.*

Proof. In [20], Trotter proved the equivalence between conditions (a) and (b). So we simply have to show the equivalence between conditions (b) and (c).

(c) $\Rightarrow$ (b) Suppose that (b) does not hold, i.e., H contains an odd cycle of length at least five. Then this cycle forms a connected partial subgraph H' of H which clearly satisfies none of the properties (i), (ii), (iii), (iv), a contradiction.

(b) $\Rightarrow$ (c) Let H' be any connected partial subgraph of H, and let ω be the size of a maximum clique of H'.

If $\omega \leqslant 2$, then H' contains no cycle of length 3. By the hypothesis, H' does not contain any odd cycle of length at least 5. We obtain that H' contains no odd cycle. Thus H' is bipartite and property (i) holds.

If $\omega = 3$, let $\{a, b, x_1\}$be any triangle of H'. We first note that

$$\text{At most one edge of the triangle may lie in other triangles.} \tag{1}$$

Let us assume that (1) does not hold. Then, without loss of generality, there exist a common neighbor u of a and b with $u \neq x_1$ and a common neighbor v of a and x_1 with $v \neq b$. If $u = v$ then $\{a, b, x_1, u\}$ is a clique of size 4, a contradiction to $\omega = 3$. If $u \neq v$ then (u, a, v, x_1, b, u) is a cycle of length 5, a contradiction to (b). So (1) is proved.

We assume now that ab is the only edge of $\{a, b, x_1\}$ that may lie in several triangles. Let $x_1, x_2, ..., x_p$ be all the vertices that are adjacent to both a and b. The set $\{x_1, ..., x_p\}$ must be independent, for otherwise there exist two adjacent vertices x_i and x_j, and then $\{a, b, x_i, x_j\}$ induces a clique of size 4, a contradiction to $\omega = 3$. If H' has no other vertices, then property (iii) holds. Else, by the connectedness of H', we can assume the existence of a vertex c different from $a, b, x_1, ..., x_p$ and adjacent to (without loss of generality) either a or x_1.

Suppose that c is adjacent to a. Note that c is not adjacent to b, for otherwise $c = x_i$ should hold for some i, contradicting the choice of c; and that c is not adjacent to any of $x_1, ..., x_p$, for otherwise some triangle would have two edges lying in other triangles, a contradiction to (1), which must hold for any triangle. We claim that c and $\{b, x_1, ..., x_p\}$ are not in the same connected component of $H' - a$. Otherwise, there must exist a chordless path P of length at least 2, in $H' - a$, connecting c to either b or some x_i. According to the parity of P, and to whether the extremity of P is b or some x_i, it is easy to see that either $P + ba + ac$, or $P + bx_1 + x_1a + ac$, or $P + x_ia + ac$, or $P + x_ib + ba + ac$ is an odd cycle of length at least 5 in H', a contradiction to the hypothesis. Thus a is a cut-vertex of H', and property (iv) holds.

In the case where c is adjacent to x_1, we can show that x_1 is a cut-vertex of H'. (The details are similar to those of the preceding paragraph and are omitted.)

If $\omega = 4$, let $\{a, b, c, d\}$ be a clique of size 4 of H'. If H' has no other vertices, then property (ii) holds. Else, by the connectedness of H', we may assume the existence of a neighbor x of a with $x \notin \{b, c, d\}$. If x is adjacent to another vertex of $\{b, c, d\}$, say b, then (x, a, c, d, b, x) is a cycle of length 5, a contradiction to (b). If a is a cut-vertex of H' then (iv) holds. If a is not a cut-vertex of H', then there must exist a path Q of length at least 2, connecting x to (without loss of generality) b in $H' - a$. Then, either $Q + ba + ax$ or $Q + bc + ca + ax$ is an odd cycle of length at least 5, a contradiction to (b).

If $\omega \geqslant 5$, then H' possesses a clique of size 5, and this clique contains a cycle of length 5, a contradiction to (b). ▮

THEOREM 3 (Berge [1]). *Let G be a graph with a clique-cutset C. Let $R_1, ..., R_q$ be the connected components of $G - C$. Then G is perfect if and only if the subgraph of G induced by $C \cup R_i$ is perfect for all $i = 1, ..., q$.* ▮

Before we give the proof of Theorem 1, we need to introduce some terminology and a technical lemma.

A *diamond* is a simple graph with four vertices and degree sequence (3, 3, 2, 2). The edge between the two vertices of degree 3 is called the *central edge* of the diamond.

A graph is *strongly perfect* [5] if every induced subgraph F contains an independent set that meets every maximal clique of F.

THE REORIENTATION LEMMA. *Let D be a normal orientation of a graph G. Suppose that e is not the central edge of a diamond in G, and that e is symmetrically directed in D. Then it is possible to delete one of the two arcs of D corresponding to e, in such a way that the resulting digraph is a normal orientation of G.*

Proof. Let $e = xy$. Because e is not the central edge of a diamond, there exists a unique maximal clique A of G containing both x and y. Since D is normal, there exist a sink a_1 of A in D, then a sink a_2 of $A - a_1$, etc. So we have $A = \{a_1, ..., a_l\}$ with $l = |A|$, and a_i is a sink of $\{a_i, ..., a_l\}$ in D, for all $i = 1, ..., l$. Without loss of generality, $x = a_i$ and $y = a_j$ with $i < j$.

Let D' be the digraph obtained by removing the arc $xy = a_ia_j$ from D. It is clear that D' is an orientation of G. We show that D' is normal. Consider any clique C of D'. If $\{x, y\} \not\subseteq C$, then C is unaffected by the removal of xy, and thus C has at least one sink in D' (the same as in D). If $\{x, y\} \subseteq C$, then $C \subseteq A$. In this case, let $k = \min\{p \mid a_p \in C\}$. By the definition of k, we obtain that a_k is a sink of C in D'. Thus every clique of D' has a sink, and D' is normal. ∎

Proof of Theorem 1

"If" Part. We claim that if $G = L(H)$ is solvable then G does not contain any odd chordless cycle of length at least 5.

Suppose, to the contrary, that there is such a cycle Z. We direct the edges of Z cyclically and with no smmetric arcs; we orient $V(G) - Z$ acyclically; finally we direct all the remaining edges from $V(G) - Z$ to Z. This yields an orientation D of G.

First we show that

$$D \text{ is a normal orientation of } G. \tag{2}$$

To prove (2), consider a clique C of D. If $C \cap Z = \emptyset$, then C has a sink since $D - Z$ is acyclic. If $C \cap Z \neq \emptyset$, then $|C \cap Z| \leqslant 2$ and it is easily seen that any sink of $C \cap Z$ is a sink of C.

Next we show that

$$D \text{ has no kernel.} \tag{3}$$

Suppose on the contrary that there exists a kernel K of D. Since no vertex of Z has any successor outside Z, it must be that $K \cap Z$ is a kernel of the subdigraph $D(Z)$ of D induced by Z. However, an odd directed cycle with no symmetric arcs has no kernel. This leads to a contradiction.

Now (2) and (3) contradict the hypothesis that G is solvable. So we must conclude that H contains no odd cycle of length at least 5. By Theorem 2, this implies that $G = L(H)$ is perfect.

"Only if" Part. We assume that G is a perfect line-graph and show that G is solvable by induction on the number n of its vertices. The fact is trivial for small n. Since G is perfect, by Theorem 2 we know that H satisfies condition (c), and thus we can break up the proof into four cases.

Case (i) *Graph H is bipartite.* Let D be any normal orientation of G. We prove that D has a kernel by induction on n. For fixed n, we also use a secondary induction on the number of arcs of D. The conclusion is trivial for small numbers.

Since H is a bipartite graph, we can paint its vertices with two colors, say turquoise and mauve, such that no two vertices of the same color are adjacent. Let $t_1, ..., t_r$ be the turquoise vertices. In G, let T_i $(i = 1, ..., r)$ be the clique formed by the vertices corresponding to the edges incident to t_i.

Similarly, let $m_1, ..., m_s$ be the mauve vertices, and M_j ($j = 1, ..., s$) be the clique of G formed by the vertices corresponding to the edges of H incident to m_j. Hence every vertex of G lies in exactly one turquoise clique and one mauve clique. We call any set $T_i \cap M_j$ that is not empty an *atom*. Thus each atom corresponds to a maximal set (possibly of size 1) of multiple edges of H, and the atoms form a partition of $V(G)$. We may assume that

$$\text{No edge between two atoms is symmetrically directed.} \tag{4}$$

To justify (4), first note that, if an edge of G is the central edge of a diamond, its endpoints must represent multiple edges of H, and so they must lie in the same atom. Consequently, an edge of G which is between two atoms is not the central edge of a diamond. Now if (4) does not hold, there exists an edge e between two atoms which is directed symmetrically in D. By the Reorientation Lemma, we know that we can delete one of the two arcs corresponding to e, in such a way that the resulting digraph D' is still a normal orientation of G. By the induction hypothesis on the number of arcs of D, we know that D' has a kernel K. It is clear that K is also a kernel of D. So we can assume that (4) holds.

Let T be any turquoise clique. Since D is normal, there is a sink x of T. Let M be the mauve clique containing x. We can assume that

$$\text{Vertex } x \text{ is a source of } M. \tag{5}$$

If (5) fails, let y be a source of M. Thus $y \neq x$. By the induction hypothesis, $D - y$ has a kernel K. If $x \in K$, then K is a kernel of D because x is a successor of y. If $x \notin K$, then x must have a successor k in K. However, since x is a sink of T and by (4), all successors of x must be in M. In particular, $k \in M$. Since y is a source of M, k is a successor of y, and thus K is a kernel of D. So we can assume that (5) holds.

Let K be the set obtained by picking one sink x_i in each turquoise clique T_i ($i = 1, ..., r$). We will show that K is a kernel of D. We know that every vertex of D lies in a turquoise clique, and it follows from the definition of K that K is absorbant. It remains to show that K is independent. Suppose that it is not: there exist two adjacent vertices x_i and x_j in K. Since they are adjacent and not in the same turquoise clique, they must lie in the same mauve clique M. By (5), since x_i is a sink of T_i, it must be that x_i is a source of M. Similarly, x_j must be a source of M. Consequently, the edge $x_i x_j$ is directed symmetrically in D. However, x_i lies in the atom $T_i \cap M$, whereas x_j lies in the atom $T_j \cap M$. This contradicts (4). This completes the proof for case (i).

Cases (ii) *and* (iii). In case (ii), H is a clique with four vertices. We let A be the set of edges incident to a given vertex x of H, and B be the set of edges not incident to x. In case (iii), we let A be the set of all edges incident to vertex a and B be the set of all edges incident to vertex b. In either case, both A and B induce a clique in G, and $V(G) = A \cup B$. Thus G is the complement of a bipartite graph. Since any bipartite graph is strongly perfect, the desired conclusion follows from Theorem 4 below.

Case (iv). *Graph H has a cut-vertex.* If every component of $H - a$ consists of one single vertex, then all edges of H are incident to a, and consequently G is a clique. In this case the conclusion is immediate.

Else, let H' be a component of $H - a$ with several vertices. It is clear that the set of all edges that are incident to a and to a vertex of H' form a clique-cutset C in G. Let D be any normal orientation of G. We can assume by induction on the number of vertices of D that every proper induced subdigraph of D has a kernel. It follows from Theorem 5 below and from the induction hypothesis that D has a kernel. Thus G is solvable. ∎

THEOREM 4 [10]. *The complement of any strongly perfect graph is solvable.*

THEOREM 5 [17]. *Let D be a directed graph with a clique-cutset C. Let $R_1, ..., R_q$ be the connected components of $G - C$. If, for all i in $\{1, ..., q\}$, any induced subdigraph of $D(C \cup R_i)$ has a kernel, then D has a kernel.*

Theorem 1 has a corollary concerning the famous Stable Marriage Problem [15]. Consider a heterosexual society, with an equal number of men and women, in which one would like to marry each man with exactly one woman. Every person ranks (i.e., makes a linear but not necessarily strict ordering on the set of) all persons of the other sex. Moreover one would like the marriage to be *stable*, which means that there should be no couple whose members would both prefer to be matched together rather than matched with their respective mates by the marriage (an "unstable couple").

We represent this society by the complete bipartite graph $B = (W, M; W \times M)$, where W is the set of all women and M is the set of all men. Call L the line graph of B. Each woman w corresponds to a clique C_w of L consisting of all vertices (m, w), $m \in M$; each man m corresponds to a clique C_m of L consisting of all vertices (m, w), $w \in W$. Now we orient L as follows: for each woman we orient the edges within C_w according to the preference of the woman, i.e., if she prefers man m_1 to man m_2, we orient the edge from vertex wm_2 to vertex wm_1, and so on; if she ranks m_1 and m_2 equaly we orient the ede (wm_1, wm_2) symmetrically; we do the same for each clique C_m, $m \in M$.

Consider a clique C of L. Clearly C is included in a C_x for some $x \in W \cup M$, and it is easy to see that C admits as a sink the vertex xy such that y is the person preferred by x among all persons (other than x)

incident to an edge represented in C. Hence L is normal, and by Theorem 1, L possesses a kernel K. Now observe that K is a maximal matching of B, which in turn implies that K is maximum since B is a complete bipartite graph; so K matches every person. Finally, asume that there exists an unstable couple $\{w, m\}$ for K; then it is easy to see that in L the vertex wm is not in K and has no successor in K, contradicting the fact that K must be absorbant. So K is a perfect stable mariage.

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